THE ROTATION OF THE EARTH, AND THE SECULAR ACCELERATIONS OF THE SUN, MOON AND PLANETS

H. Spencer Jones, F.R.S., Astronomer Royal

1. It has been fully established by the researches of various investigators that there are fluctuations in the longitudes of the Sun, Mercury and Venus, which run closely parallel to the fluctuations in the longitude of the Moon. The fluctuations in the longitudes of these several bodies have therefore been attributed to a common cause, a variation of the adopted unit of time provided by the rotation of the Earth.

Variations in the rotation of the Earth have been attributed to two causes :

(*a*) Tidal friction, particularly in enclosed or narrow seas.

(*b*) Changes in the moment of inertia of the Earth, as the result of some redistribution of matter within the Earth.

Tidal friction can cause only a retardation of the rotation, never an acceleration. The portions of the secular accelerations of the longitudes of the Sun and Moon, not explained by gravitational theory, are attributed to the slowing down of the Earth by tidal friction.

Changes in the moment of inertia of the Earth may result in acceleration or retardation of the rotation, according as the moment of inertia is decreased or increased. The irregular fluctuations in longitude are attributed to this cause.

The effects produced by the two causes differ in one respect. If there is a change in the moment of inertia of the Earth, the effects on the longitudes of the Sun, Moon and planets, when expressed in time are identical or, in other words, the effects on the longitudes, when expressed in arc, are proportional to the mean motions. Associated with the retardation of the Earth by tidal friction there is a change of the angular momentum of the Earth, which must be compensated by a corresponding change in the angular momentum of the Moon's orbital rotation (the angular momentum associated with the axial rotation of the Moon can be neglected). Tidal friction therefore produces a change in the true mean motion of the Moon. The effects on the longitudes of the planets will be identical in time, or, otherwise expressed, the effects in angle will be proportional to the mean motions ; but the effect on the longitude of the Moon, bound up as it is with changes in the shape of the Moon's orbit, cannot be predicted by theory.

These expectations, though not so fully substantiated by observation as is desirable, have been generally confirmed with one exception, referred to in section 3.

2. For the purpose of discussion of the observational material, it is

necessary to formulate an expression for the fluctuation of the longitude of the Moon, or, in other words, for the difference between the true longitude and the longitude according to pure gravitational theory. In Brown's Tables of the Moon, the great empirical term of Newcomb was included in order to secure approximate agreement with observations and the adopted value of the secular acceleration of the Moon was the theoretical value, produced by the attractions of the planets. Brown considered the total secular acceleration was not sufficiently well determined by observation to be used in his tables.

The investigations of Fotheringham, summarised and rediscussed by de Sitter, lead to a value of $5''\cdot22$ as best representing the average value of the residual secular acceleration of the Moon, not accounted for by gravitational theory, over a period of about two thousand years. The question whether this value can be accepted as representing modern observations will be discussed subsequently.

We write, therefore,

$$\begin{aligned} B &= \text{Observed Longitude} - C, \\ C &= \text{Brown's Tables} - 10''\cdot71 \sin(140^\circ\cdot0T + 240^\circ\cdot7) \\ &\quad + 5''\cdot22T^2 + 12''\cdot96T + 4''\cdot65, \end{aligned}$$

and define the quantity B as the *fluctuation* in the Moon's longitude. We have removed the great empirical term, applied a correction to reduce from the theoretical secular acceleration adopted by Brown to the observed value and made consequential corrections to the mean motion and longitude at epoch (which are purely observational quantities) in order to secure close agreement with modern observations.

It will be noted that we have separated the secular acceleration term from the fluctuation. We have, in effect, represented the difference between observation and pure gravitational theory by an expression of the form

$$\alpha + \beta T + 5''\cdot22T^2 + B.$$

The question whether the fluctuations and the secular acceleration can be uniquely separated will be discussed subsequently.

3. We will assume that the correction required by the longitude of the Sun according to Newcomb's tables may be expressed in the form

$$\Delta L = a + bT + cT^2 + Q \cdot \frac{n_0}{n} B$$

and by the planets Mercury and Venus to be

$$\Delta l_i = a_i + b_i T + c_i T^2 + Q \cdot \frac{n_i}{n} B,$$

where $i = 1$ for Mercury and $= 2$ for Venus. n denotes the mean motion of the Moon, n_0, n_1, n_2 denote the mean motions of the Sun, Mercury and Venus respectively. T is expressed throughout in centuries from 1900·0.

The quantity Q was introduced by de Sitter * to take account of the possibility that the terms involving B in the longitudes of Moon, Sun and planets were not strictly proportional to the mean motions and it was not assumed that Q had the same value for these bodies. The discussions by de Sitter and the present author † agree in establishing that, within the limits of observational error Q has the same value for the Sun, Mercury and Venus. The value given by de Sitter's discussion was 1·25; the value found by the present author was 1·19. The observational data used for the Sun, Venus and Mercury were essentially the same in the two discussions, but de Sitter used the results derived by Newcomb from eclipses and occultations up to 1835 and the Greenwich meridian observations after that date, whereas my discussion was based entirely on my revision of Newcomb's discussion of occultation observations, supplemented by more recent data.

There are great difficulties in interpreting a value of Q that is greater than unity and de Sitter's suggested explanation, involving sudden and very great changes in tidal friction, is artificial and not convincing. The main purpose of the present discussion, which has used additional observational material, and has proceeded along somewhat different lines was the reinvestigation of the value of Q. The result of this discussion is to show that Q has the value unity within the limits of probable error.

The previous discussions have also shown that within the limits of observational error we may write

$$c_1 = n_1 c/n_0 \quad \text{and} \quad c_2 = n_2 c/n_0.$$

These relationships, implying that the secular accelerations are proportional to the mean motions, are only established for the Sun and Mercury; for Venus the results that are available for use and sufficiently free from accidental error do not cover a time long enough to determine the secular acceleration of Venus with much accuracy. The Venus observations do not, however, contradict the relationship.

4. The previous discussions by de Sitter and myself were based on the following data: for the Sun, the data given by me ‡ for the period 1835 to 1923, which de Sitter supplemented by data for the period 1750 to 1834, taken from the paper by E. W. Brown §; these early observations were so erratic that they are not of much value and may seriously impair the results. For Venus, the data given by me ‡ for the period 1836 to 1923. For Mercury, the observations of the transits across the Sun, discussed by Newcomb ‖ and rediscussed by Innes ¶ were used; these extend from 1677 to 1924.

The longitude errors of the Sun that were used were based entirely on observations of right ascension. These observations are peculiarly liable to systematic errors, depending on the observer and the method of observation. The observations at Greenwich, for instance, in which several

* *B.A.N.*, 4, 21, 1927.

† *Cape Annals*, **13**, part 3, 1932 (see § vii).

‡ *M.N.*, **87**, 4, 1926.

§ *Transactions of Yale University Observatory*, **3**, part 6, 1926.

‖ *Astr. Papers American Ephemeris*, **1**, part vi, 363, 1882.

¶ *Union Observatory Circular*, No. 65, 1925.

observers participate, show some considerable personal differences, which do not necessarily remain constant for long periods of time; changes are necessarily introduced by changes in the rota of observers. The change from the eye-and-ear to the tapping method of recording, and from the tapping method to the use of the travelling wire micrometer involve discontinuities, which are not easy to assign accurately. It is accordingly impossible to be certain that the corrections to the Sun's longitude from Newcomb's tables, as inferred from observations of right ascension, are on a uniform basis.

The data for the Sun's longitude are specially important, on the other hand, for the present purpose, because they enter into the reduction of the observations of Venus and Mercury. In the reduction of the Venus observations, corrections to the Sun's tabular longitude, read off from smooth curves drawn to represent as closely as possible the observed values, were used; any systematic errors in the observations of the Sun must consequently affect the observations of Venus. The same procedure was not possible in dealing with the observations of the transits of Mercury, because these cover a much longer period than the available Sun observations. The expression derived for the representation of the Sun observations was therefore extrapolated for the much longer period of the transit of Mercury observations. These are therefore also affected by any systematic errors in the Sun observations, with the possibility that by extrapolating backwards in time their influence may have been materially increased.

Minor defects in treatment were the use of smoothed values for the longitudes of the Sun and Venus; the smoothing may introduce systematic effects. Equal weight was given to each value read off from the smoothed curve drawn to represent the observations; the earlier observations were therefore considerably overweighted.

5. The longitude of the Sun can be derived from observations of either declination or right ascension; declination observations, in the case of the Sun, are far more likely to be free from systematic errors than are the observations of right ascension. Though the correction to the Sun's tabular longitude can be inferred with much greater weight from right ascension observations than from declination observations, the declination observations are likely to give a much truer representation over a long period of time of the changes in the correction. The reduction in weight, as inferred from the internal accordance of the observations during a limited period, is more than counterbalanced by the greater freedom from errors of a systematic nature. In the present discussion, observations of declination have been utilised and have been accepted as likely to provide the best representation of the Sun's longitude. The smaller systematic error of the declination observations is supported by the fact that the longitudes deduced from the early observations of declination do not show the large and irregular variations that characterise the longitudes deduced from the observations, during the same period, of the right ascensions. It has been possible to incorporate the observations of declination back to the year 1750 with substantial advantage.

It was also decided not to derive longitudes of Venus and Mercury by using either smoothed values of the Sun's longitude or a representation of the observed longitudes by a formula, but to make direct use of the quantities actually furnished by the observations.

The observations of Venus give directly the values of $\Delta l_2 - \Delta L$, *i.e.* the difference of the longitude corrections of Venus and the Sun. The observations of the transits of Mercury give the quantities denoted by V and W by Newcomb *, viz.:

November transits : $V = 1{\cdot}487\Delta l_1 - 1{\cdot}01\Delta L +$ corrections to other elements.

May transits : $W = 0{\cdot}716\Delta l_1 - 0{\cdot}97\Delta L +$ corrections to other elements.

The observations were reduced in the following way. It is assumed that the corrections to the longitudes of the Sun, Mercury and Venus may be represented by the following expressions :—

$$\Delta L = a + bT + cT^2 + Q(\cdot 0747)B, \tag{1}$$

$$\Delta l_1 = a' + b'T + 4{\cdot}15cT^2 + Q(\cdot 310)B, \tag{2}$$

$$\Delta l_2 = a'' + b''T + 1{\cdot}63cT^2 + Q(\cdot 122)B. \tag{3}$$

The ratios of the mean motions of Mercury and Venus to the mean motion of the Sun are respectively 4·15 and 1·63.

By this representation it is assumed that the secular accelerations of the Sun and planets, arising from the retardation of the Earth by tidal friction, are proportional to their respective mean motions. No assumption is made about the ratios of these secular accelerations to the secular acceleration of the Moon. It is also assumed that the fluctuations in the longitude of the Moon are reflected in the longitudes of the Sun and planets proportionally to their mean motions. If, as is to be expected on theoretical grounds, these fluctuations are due to change of moment of inertia of the Earth, the ratio between the effects on the longitudes of Sun and Moon should be proportional to the mean motions ; or, since the ratio of the mean motions is ·0747, we should expect that Q would have the value unity.

Observations give the following expressions :—

Mercury : November Transits.—The corrections to the other elements of the orbit of Mercury may be assumed to be represented by an expression of the form $x + yT$. Hence we may write

$$V = a_1 + b_1 T + 5{\cdot}16cT^2 + Q(\cdot 385)B. \tag{4}$$

Mercury : May Transits.—In a similar way we may write

$$W = a_2 + b_2 T + 1{\cdot}97cT^2 + Q(\cdot 147)B. \tag{5}$$

* *Astr. Papers American Ephemeris*, **1**, part vi, 447, 1882.

Venus : Observations give the quantity $\Delta l_2 - \Delta L$ which we may write in the form

$$\Delta l_2 - \Delta L = a_3 + b_3 T + 0{\cdot}63 c T^2 + Q({\cdot}0471)B. \qquad (6)$$

It will be seen that the Venus observations are of much less value than the observations of Mercury or the Sun for the determination of c and Q.

The observations of Sun, Mercury and Venus were combined for the determination of c and Q.

6. *The Observational Material.* (a) *The Fluctuations in the Moon's Longitude.*—The values of the fluctuations, derived in *Cape Annals*, **13**, part 3 (Table VI, p. 31) have been used. These have been supplemented for recent years by the results derived from the errors of the Moon's longitude according to Brown's tables, as deduced from extensive observations of occultations. The reductions of these occultations have been published by Brown and Brouwer in various numbers of the *Astronomical Journal.* For convenience the fluctuations for the entire period covered by the observations that have been used are summarised in Table I.

TABLE I

Fluctuations in the Moon's Mean Longitude

Date	B	Date	B	Date	B
	″		″		″
1681·0	−12·72	1837·4	+ 4·91	1900·5	−15·87
1710·0	− 3·92	43·1	+ 4·31	03·5	−14·50
27·0	+ 2·15	48·8	+ 3·97	06·5	−13·43
37·0	+ 5·97	52·5	+ 3·37	09·5	−12·78
47·0	+ 8·49	57·5	+ 2·40	12·5	−11·62
55·0	+10·34	62·5	+ 0·91	15·5	−10·35
71·0	+13·54	67·5	− 1·57	18·5	−10·20
85·0	+14·84	72·5	− 6·38	21·5	−10·18
92·0	+14·53	77·5	− 9·38	24·5	−11·82
1801·5	+13·09	82·5	−11·31	26·5	−12·11
09·5	+11·80	87·5	−13·05	28·5	−12·90
13·0	+11·28	91·5	−14·34	30·5	−13·83
21·8	+10·02	94·5	−15·23	32·5	−14·81
31·5	+ 6·85	97·5	−15·99	34·5	−15·98
				36·5	−16·48

(b) *The Sun : Declinations.*—The material used for the Sun for the observations prior to 1900 was taken from the data given by Newcomb *, supplemented by the observations made by Hornsby at the Radcliffe Observatory from 1774 to 1798 † and the Greenwich Observations for 1893 to 1900. Corrections to reduce from Leverrier's tables to Newcomb's tables were applied, the data for this purpose being extracted from papers by Ross ‡ and myself.§

* *Astronomical Constants*, 30–32, 1895.

† Knox-Shaw, Jackson and Robinson, *Hornsby's Meridian Observations, 1774–1798*, 100–104, 1932.

‡ *Astronomical Journal*, **29**, 153, 1916.

§ *M.N.*, **86**, 428, 1926 ; *Cape Annals*, **13**, part 3, 33–34, 1932.

After 1900, the mean annual results given by the Greenwich observations up to the year 1936 were combined with the data for other observatories summarised by Morgan and Scott.*

The Greenwich series of observations made with the Airy transit circle was adopted as a basis and systematic corrections were applied to the older series of observations by comparisons of overlaps to reduce them to the same basis. The observations were weighted, an attempt being made to assign a series of weights such that unit weight corresponded to a probable error of 0″·50. The adopted mean values and weights are given in Table II, which includes also information about the observations included in each mean value.

TABLE II

Sun's Longitude (Observed minus Newcomb) from Declination

Mean Date	Correction to Tables	Weight	Observations	*B*/10	Residual (O − C)	Fluctuation
	″			″	″	″
1761·3	+ ·31	5	Gr	+1·33	+·09	+14·5
77·9	− ·01	9	Gr, H	+1·44	−·30	+10·5
89·8	+ ·32	6	Gr, H	+1·45	+·02	+14·7
99·0	+ ·68	6	Gr, H	+1·34	+·43	+19·2
1811·4	+ ·05	11	Gr, Pa	+1·15	−·13	+ 9·8
20·5	− ·49	9	Gr, Pa, Kb	+1·03	−·68	+ 1·2
25·8	+ ·22	15	Gr, Pa, Kb, Dp	+ ·87	+·09	+ 9·9
31·7	+ ·31	13	Gr, Kb, Dp	+ ·67	+·25	+10·0
37·8	− ·03	15	Gr, Pa, Dp, Ca	+ ·49	−·04	+ 4·4
42·9	+ ·15	17	Gr, Kb, Ca, Pu, Ox	+ ·43	+·13	+ 6·0
48·9	+ ·22	29	Gr, Pa, Ca, Pu, Ox, Wa	+ ·39	+·12	+ 5·5
56·7	+ ·12	15	Gr, Pa, Ca	+ ·24	−·01	+ 2·3
63·3	+ ·22	24	Gr, Pa, Pu, Ox, Wa	− ·06	+·20	+ 2·1
68·9	+ ·12	29	Gr, Pa, Pu, Ox, Wa, Lei	− ·26	+·11	− 1·1
74·6	− ·40	26	Gr, Pa, Ox, Wa, Lei	− ·79	−·14	− 9·8
81·1	− ·42	20	Gr, Pa, Ox, Wa	−1·07	−·11	−12·2
86·2	− ·29	23	Gr, Pa, Ox, Cp, Sb	−1·25	+·04	−11·9
89·5	− ·24	17	Gr, Wa, Cp	−1·37	+·07	−12·7
97·0	− ·29	15	Gr	−1·59	−·01	−16·1
1902·5	− ·16	30	Gr, Wa	−1·49	−·14	−16·7
08·2	+ ·23	90	Gr, Pu, Wa, Cp	−1·31	−·04	−13·7
13·1	+ ·60	75	Gr, Pu, Wa, Cp	−1·14	+·04	−10·9
17·0	+ ·52	37	Gr, Al	−1·03	−·25	−13·7
22·1	+ ·90	45	Gr, Wa, Be, Br	−1·05	−·03	−10·9
27·0	+1·08	37	Gr, Wa	−1·23	+·11	−10·9
30·4	+1·08	37	Gr, Wa	−1·38	+·10	−12·5
34·2	+1·05	30	Gr, Wa	−1·58	+·07	−14·9

In the above table the following abbreviations are used: Gr = Greenwich; H = Hornsby (Radcliffe); Pa = Paris; Kb = Königsberg; Dp = Dorpat; Ca = Cambridge; Pu = Pulkova; Ox = Oxford (Radcliffe); Wa = Wash-

* *Astronomical Journal*, **47**, 193, 1939.

ington; Lei = Leiden; Cp = Cape of Good Hope; Sb = Strasbourg; Al = Algiers; Be = Berlin-Babelsberg; Br = Breslau.

The fifth column of the table gives the values of $B/10$, where B is the quantity adopted to represent the fluctuation in the Moon's longitude. The residuals from the finally adopted solution are given in the sixth column. The last column of the table will be explained later.

(c) *The Sun: Right Ascensions.*—The data used in my previous discussion * were utilised, but mean values for each few years were taken. The Greenwich results have been extended to the year 1936 and the data for other observatories, from 1900 onwards, given by Morgan and Scott †, have been incorporated. The observations have been reduced to the basis of the Greenwich observations with the Airy transit circle, using the hand-tapper. A discontinuity of $0^s.030$ in the Greenwich results with the introduction of the travelling wire has been adopted.

The mean values together with the adopted weights, assigned on the basis of unit weight corresponding to a probable error of $0''.50$, are given in Table III together with the observatories whose observations have been incorporated in each mean value. The table is arranged similarly to Table II.

TABLE III

Sun's Longitude (Observed minus Newcomb) from Right Ascensions

Mean Date	Correction to Tables	Weight	Observations	$B/10$	Residual (O − C)	Fluctuation
	″			″	″	″
1839·0	+ ·34	7	Gr, Kb, Dp, Ca	+ ·46	−·10	+ 3·2
47·4	+ ·54	7	Gr, Ca, Pu, Wa	+ ·41	+·02	+ 4·4
56·5	+ ·51	5	Gr, Ca	+ ·26	−·02	+ 2·3
63·2	+ ·33	7	Gr, Wa	+ ·05	−·17	− 1·7
68·3	+ ·29	14	Gr, Pu, Wa	− ·19	−·12	− 3·5
73·5	+ ·37	5	Gr	− ·72	+·25	− 3·9
78·3	+ ·39	9	Gr, Wa	− ·97	+·35	− 5·0
83·0	+ ·20	5	Gr	−1·15	+·20	− 8·8
87·2	− ·35	16	Gr, Cp, Wa, Sb	−1·29	−·35	−17·6
92·5	− ·18	7	Gr, Pa	−1·46	−·18	−17·0
96·5	+ ·23	13	Gr, Wa	−1·58	+·22	−12·9
99·5	+ ·13	13	Gr, Pa, Wa	−1·61	+·07	−15·1
1902·5	+ ·12	16	Gr, Pa, Wa, Pu	−1·49	−·12	−16·5
05·7	+ ·24	16	Gr, Pa, Wa, Pu	−1·36	−·17	−15·9
09·7	+ ·66	27	Gr, Wa, Cp, Pu	−1·27	+·06	−11·9
14·9	+1·04	45	Gr, Wa, Cp, Pu, Al	−1·06	+·12	− 9·0
19·7	+1·14	32	Gr, Wa	−1·02	+·04	− 9·6
24·0	+1·25	27	Gr, Wa	−1·16	+·12	−10·1
28·0	+1·24	18	Gr	−1·27	+·06	−11·9
32·2	+1·01	28	Gr, Wa	−1·45	−·19	−17·0
35·3	+1·03	23	Gr, Wa	−1·62	−·14	−18·1

* *Cape Annals*, **13**, part 3, 1932.

† *Astronomical Journal*, **47**, 193 (Table I), 1939.

(d) *Transits of Mercury*.—The data given by the rediscussion of the observations of the transits of Mercury by Innes, as summarised by de Sitter *, have been adopted without alteration. The only addition is the inclusion of data for the 1927 transit, compiled from the mean of all the observations that have been published. The transits of 1782 November and 1740 May, which are badly discordant, have been discarded.

The data are summarised in Table IV, which gives the observed values of V and W from the November and May transits respectively, the relative weights on the same basis as for the Sun observations, the quantity $B/10$, which is proportional to the fluctuations in the Moon's longitude, and the residuals from the final solution. The last column will be explained later.

TABLE IV

Transits of Mercury

(*a*) November Transits

Date	V	Weight	$B/10$	Residual (O − C)	Fluctuation
	″		″	″	″
1677·9	−2·57	0·1	−1·36	+1·11	−10·7
90·9	−2·32	0·1	−1·10	+1·83	− 6·2
97·9	−7·09	0·3	−0·91	−2·96	−16·8
1723·9	−1·10	1·0	+0·09	+1·24	+ 4·1
36·9	−0·42	1·7	+0·60	+0·67	+ 7·7
43·9	−0·86	1·9	+0·77	−0·14	+ 7·3
56·9	−0·11	0·2	+1·08	−0·21	+10·3
69·9	−0·50	0·9	+1·29	−1·20	+ 9·8
89·9	+1·38	1·7	+1·47	−0·16	+14·3
1802·9	+0·34	0·6	+1·29	−0·88	+10·6
22·9	+0·76	1·1	+0·98	−0·23	+ 9·2
48·9	+1·35	2·7	+0·46	+0·29	+ 5·4
61·9	+1·82	2·1	+0·13	+0·64	+ 3·0
68·9	−0·08	3·5	−0·25	−0·61	− 4·1
81·9	−1·23	4·0	−1·07	−0·22	−11·3
94·9	−0·47	4·3	−1·53	+0·41	−14·2
1907·9	+2·10	4·1	−1·31	−0·01	−13·1
14·9	+4·14	4·5	−1·06	−0·11	−10·9
27·9	+5·92	5·4	−1·27	+0·02	−12·7

(*b*) May Transits

Date	W	Weight	$B/10$	Residual (O − C)	Fluctuation
	″		″	″	″
1753·4	−0·78	2·0	+1·00	−0·18	+ 8·8
86·4	−0·18	7·5	+1·48	−0·45	+11·8
99·4	+0·32	7·0	+1·34	+0·05	+13·7
1832·4	+0·42	11·5	+0·65	+0·24	+ 8·2
45·4	+0·56	13·0	+0·42	+0·22	+ 5·7
78·4	−0·06	12·5	−0·97	+0·01	− 9·7
91·4	−0·05	12·5	−1·43	−0·10	−15·0
1924·4	+2·59	7·0	−1·18	−0·21	−13·2

* *B.A.N.*, 4, 31, 1927.

(e) *Venus*.—The data have been taken from my previous paper *, supplemented by more recent Greenwich observations. For the purpose of deriving longitude errors of Venus the final adopted smoothed errors of the Sun's longitude were used. For the present discussion the data are given in the form actually derived in the first instance, *i.e.* the values of $\Delta l_2 - \Delta L$ are tabulated. The material is summarised in Table V.

TABLE V

Longitude Difference (Venus minus Sun): Observed minus Newcomb's Tables

Date	Long. Diff.	Weight	$B/10$	Residual	Fluctuation
	″		″	″	″
1839·0	− ·67	2	+ ·45	− ·49	− 5·9
45·4	+ ·13	3	+ ·42	+ ·21	+ 8·7
51·8	+ ·31	5	+ ·35	+ ·27	+ 9·1
58·2	+ ·41	5	+ ·23	+ ·31	+ 8·9
64·5	+ ·30	5	− ·02	+ ·20	+ 4·0
70·9	− ·14	5	− ·43	− ·21	− 8·7
77·3	+ ·25	5	− ·93	+ ·29	− 3·2
83·7	− ·45	6	− 1·17	− ·47	− 21·6
90·1	− ·34	8	− 1·39	− ·41	− 22·5
96·5	− ·17	6	− 1·58	− ·30	− 22·0
1902·9	+ ·52	7	− 1·47	+ ·15	− 11·4
09·3	+ ·64	9	− 1·28	+ ·01	− 12·5
13·5	+ ·72	10	− 1·12	− ·12	− 13·8
16·5	+ 1·13	9	− 1·03	+ ·16	− 7·0
20·0	+ 1·41	8	− 1·02	+ ·33	− 3·2
23·6	+ 1·52	8	− 1·13	+ ·37	− 3·4
26·6	+ 1·29	9	− 1·22	+ ·10	− 10·0
29·6	+ 1·39	9	− 1·34	+ ·16	− 10·0
33·1	+ 1·06	8	− 1·52	− ·18	− 19·1
35·5	+ ·80	6	− 1·62	− ·49	− 26·5

7. *Derivation of the Constants*.—The expressions (1), (4), (5) and (6) in section 5 were adopted to represent the observed data summarised in Tables II to V. Normal equations were formed, the whole of the material being combined for the quantities c and Q. Two solutions were made: in the first the data for the longitude errors of the Sun from right ascensions were excluded; in the second they were included. The results of these solutions were as follows:—

I. *Sun's R.A.'s Excluded*

$a + 1''{\cdot}02$	$a_1 + 5''{\cdot}90$	$a_2 + 2''{\cdot}78$	$a_3 + 1''{\cdot}00$
$b + 3''{\cdot}02$	$b_1 + 16''{\cdot}26$	$b_2 + 6''{\cdot}94$	$b_3 + 2''{\cdot}72$
	$c = +1''{\cdot}25$	$Q = 1{\cdot}025.$	

* *M.N.*, **87**, 4 (Table VII), 1926.

II. *Sun's R.A.'s Included*

a, b denote the correction to the longitude at epoch and to the mean motion of the Sun derived from declination observations; a', b' the corresponding quantities derived from right ascension observations.

$a+1''\cdot04$	$a'+1''\cdot32$	$a_1+\ 6''\cdot03$	$a_2+2''\cdot85$	$a_3+1''\cdot02$
$b+3''\cdot09$	$b'+2''\cdot67$	$b_1+16''\cdot55$	$b_2+7''\cdot10$	$b_3+2''\cdot76$

$$c=+1''\cdot26 \qquad Q=+1\cdot062.$$

The probable error of the value of Q is about $\pm\cdot033$. It will be seen, therefore, that when the right ascension observations of the Sun are excluded the value of Q is unity, within an amount that is less than the probable error. When the right ascension observations of the Sun are included the value of Q is increased; it is considered that the increase has been caused by systematic errors in the right ascension observations of the Sun. It is accordingly concluded that the observations provide no justification for the assumption that Q does not have the value unity, and, moreover, as has already been mentioned, there are great difficulties in interpreting any other value.

It is accordingly assumed that we may write $Q=1$. The derived value of c is then practically unaffected by the inclusion of the right ascension observations; these contribute very little weight to the determination of c because they extend over a relatively short period.

With this assumption we obtain—

III. *Adopted Solution*

$a+1''\cdot00$	$a'+1''\cdot28$	$a_1+\ 5''\cdot81$	$a_2+2''\cdot74$	$a_3+0''\cdot98$
$b+2''\cdot97$	$b'+2''\cdot69$	$b_1+16''\cdot01$	$b_2+6''\cdot82$	$b_3+2''\cdot70$

$$c=1''\cdot23 \qquad Q=+1\cdot00.$$

With these values the following probable errors corresponding to unit weight are obtained: for the declination observations of the Sun, $\pm\ 0''\cdot49$; for the right ascension observations of the Sun, $\pm\ 0''\cdot42$; for the November transits of Mercury, $\pm\ 0''\cdot50$; for the May transits of Mercury, $\pm\ 0''\cdot43$; for the observations of Venus, $\pm\ 0''\cdot48$. As an attempt was made to assign weights on the basis that unit weight corresponded to a probable error of $\pm\ 0''\cdot50$, these values may be regarded as satisfactory.

The fluctuations given in the last column of Tables II to V were derived from this adopted solution. In the expressions (1), (4), (5) and (6) the values of the coefficients a, b and c were inserted, $Q=1$ was assumed and the value of B was inferred. If the assumptions involved in the analysis are correct, the values of B derived in this way should agree generally with those adopted for the Moon. The values are plotted in the two figures, the first of which shows the comparison for the full period covered; the second shows on a larger scale the comparison from 1835 onwards, the right ascension observations of the Sun and the observations of Venus

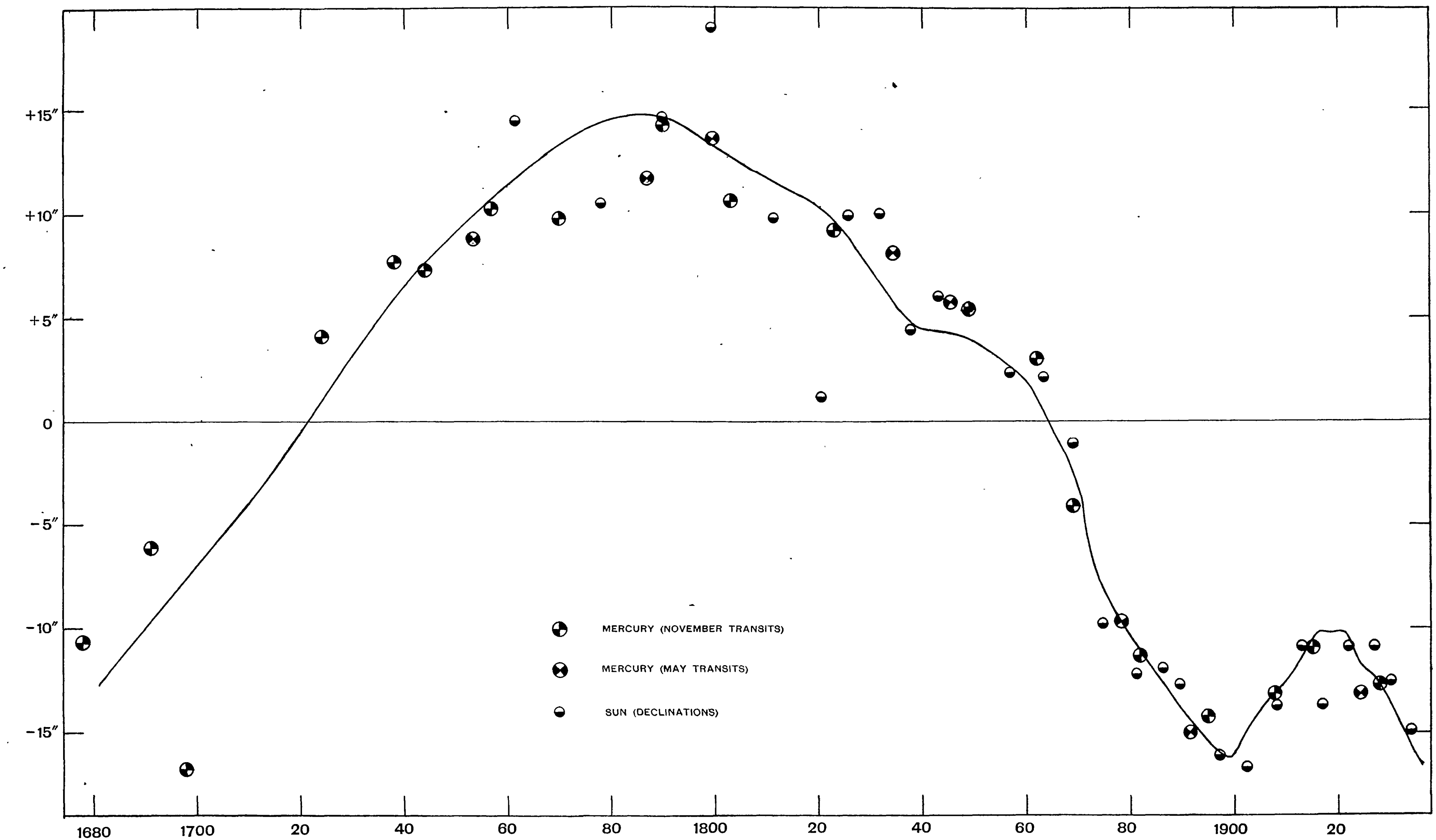

Fig. 1.—*Fluctuations in the longitude of the Moon from 1680, from observations of the Moon, Sun and Mercury.*

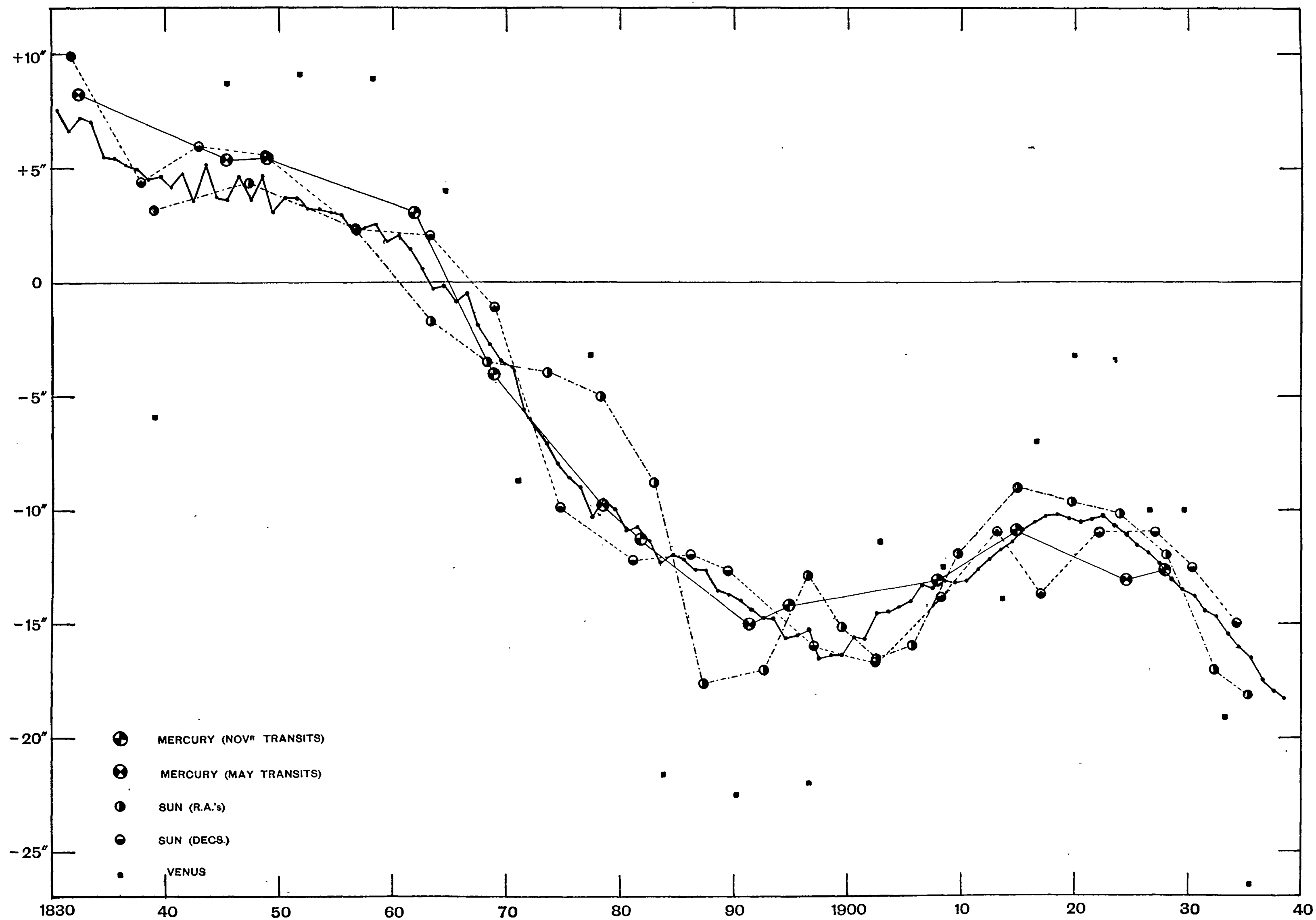

FIG. 2.—*Fluctuations in the longitude of the Moon, inferred from observations since* 1830 *of the Moon, Sun, Mercury and Venus.*

being restricted to this shorter period. The agreement is on the whole extremely good; the chief discordances are shown in the Venus observations, but this is to be anticipated because B is determined from equation (6) with very low weight. In fig. 1 a smoothed representation of the Moon's fluctuations is drawn; in fig. 2 observed values are given.

8. *The Secular Accelerations of the Sun and Moon.*—We have adopted a value of $5''.22$ for the portion of the secular acceleration of the Moon that is due to tidal friction and have derived a value of $+1''.23 \pm 0''.04$ for the secular acceleration of the Sun.

This secular acceleration has been derived from the solution that combines all the material. It is of interest to examine to what extent the values given by the different series of observations are in agreement. Assuming, as before, that $Q=1$ we obtain the following values:—

Sun	Declinations	$+1''.19 \pm 0''.11$
	Right Ascensions	$+0''.47 \pm 0''.41$
	Combined	$+1''.14 \pm 0''.11$
Mercury	November Transits	$+1''.27 \pm 0''.04$
	May	$+0''.83 \pm 0''.14$
	Combined	$+1''.24 \pm 0''.04$

The Venus observations have negligible weight for the determination of the secular acceleration.

The meridian observations of the Sun and the observations of the transits of Mercury are thus in close agreement.

From the ancient observations of solar and lunar eclipses and of occultations, as reduced and discussed by Fotheringham and collected and rediscussed by de Sitter *, the following values of the secular accelerations are derived as the most probable:—

For the Moon	$+5''.22 \pm 0''.30$,
For the Sun	$+1''.80 \pm 0''.16$.

This value for the secular acceleration of the Moon was adopted for the derivation of the fluctuations in the motion of the Moon used in the present discussion. The difference between the secular acceleration of the Sun derived from the ancient observations and that derived in the present discussion is about three times the sum of the probable errors of the two determinations. Though it is impossible to be certain, it seems probable that the difference is significant particularly in view of the close agreement between the values given by the Sun and Mercury. There is no *a priori* reason why the value derived in the present investigation should necessarily be in agreement with the value derived from the ancient eclipses; the value derived from ancient eclipses represents a mean value during the past twenty centuries, whilst the value derived from the present discussion represents the mean value during the past two centuries or so. It is possible that the effects of tidal friction and the consequent retardation of the rotation

* *B.A.N.*, 4, 21, 1927, No. 124 (see § 2).

of the Earth have not remained absolutely constant. A slow progressive change in the effects of tidal friction seems to be not improbable.

We have assumed, however, for the secular acceleration of the Moon the value derived from the ancient eclipses. If tidal friction effects do not remain constant this assumption is not correct. Some assumption must necessarily be made, because otherwise the irregular fluctuations in longitude cannot be separated from the effects of the secular acceleration. We have implicitly assumed that during the period for which fairly reliable values of the longitude of the Moon are available, which we may take to be the last two hundred and fifty years, the longitude of the Moon has fluctuated by approximately equal amounts on either side of its mean value.

If we suppose that during the period covered by the present investigation the true secular acceleration of the Moon is not $+5''{\cdot}22$, as we have supposed, but $+5''{\cdot}22+s$, then we must replace B by a quantity B', which we may define by

$$B' = B - sS,$$

where $S = T^2 + 1{\cdot}3T - 0{\cdot}3$. The quantity S, which has zero values at 1750 and 1920, is introduced instead of T^2 in order to secure agreement with modern observations; in other words, it automatically takes account of the necessary adjustments to longitude at epoch and to mean motion.

The term in T^2 and B in the Sun and planets then becomes

$$\text{For the Sun,} \qquad \frac{n_0}{n}(B - sS) + \left(c + \frac{n_0}{n}s\right)T^2,$$

$$\text{For the planets,} \qquad \frac{n_i}{n_0}\left[\frac{n_0}{n}(B - sS) + \left(c + \frac{n_0}{n}s\right)T^2\right].$$

The analysis therefore proceeds as before, but corresponding to a secular acceleration for the Moon of $+5''{\cdot}22+s$ we will derive a secular acceleration for the Sun of amount $(c + n_0 s/n)$, which is equal to $+1''{\cdot}23 + ({\cdot}0747)s$.

Now, though there is nothing inherently improbable in the supposition that the effects of tidal friction are slowly variable, it seems valid to assume that the effects on the secular accelerations of the Moon and Sun remain in a constant ratio. The value of this ratio cannot be determined theoretically, because there are changes in some of the elements of the Moon's orbit that cannot be calculated and are too small to be checked by observation. It may be assumed, therefore, that at any time the ratio of the secular accelerations of the Moon and Sun caused by tidal friction is equal to the ratio of the average values over the last two thousand years, which can be inferred from the ancient eclipse and occultation observations.

With this assumption we have the relationship

$$\frac{1{\cdot}80 \pm {\cdot}16}{5{\cdot}22 \pm {\cdot}30} = \frac{(1{\cdot}23 \pm {\cdot}04) + {\cdot}0747s}{5{\cdot}22 + s},$$

which gives $s = -2{\cdot}11 \pm {\cdot}57$.

The best values that we can assign for the secular accelerations of the

Sun and Moon at the present time (or more strictly, the best average values for the past two hundred and fifty years) are therefore

For the Moon . . . $+3''\cdot11 \pm 0''\cdot57$,
For the Sun . . . $+1''\cdot07 \pm 0''\cdot06$.

These values of the accelerations will not satisfy any of the ancient observations of eclipses and occultations, which are on the whole in very good agreement with one another in requiring appreciably larger values. There seems to be no escape from the conclusion that the effects of tidal friction are appreciably less at the present time than the average effects over the past two thousand years. There is no indication of the large and sudden fluctuations in tidal friction suggested by de Sitter *, and it is not necessary to assume that any sudden variations have taken place.

For the purpose of representing current observations by a formula, it is sufficient to retain B as defined in section 2 and the secular accelerations determined on this basis. The best possible separation of "fluctuations" from true secular acceleration is obtained, however, by adopting the smaller secular accelerations given above and assigning the remaining portion to the fluctuations.

9. *Corrections to Mean Longitudes of Sun, Mercury and Venus. The Sun.*—The declination observations give directly the correction to the true longitude of the Sun, $\Delta\lambda$. The right ascension observations give the corrections to the right ascension, $\Delta\alpha$, measured from the zero point of the clock-star system. We have obtained the following values :—

$$\Delta\lambda = +1''\cdot00 + 2''\cdot97T + 1''\cdot23T^2 + \cdot0747B$$
$$\pm \cdot02 \pm \cdot05,$$
$$\Delta\alpha = +1''\cdot28 + 2''\cdot69T + 1''\cdot23T^2 + \cdot0747B$$
$$\pm \cdot03 \pm \cdot12.$$

The difference $\Delta\lambda - \Delta\alpha = E$, the equinox correction, or correction to the zero of the clock-star system, is

$$E = -0''\cdot28 \pm 0''\cdot04 + (0''\cdot28 \pm 0''\cdot13)T.$$

The right ascension observations have been reduced to the system of the Greenwich transit instrument with observations by the hand-tapping method. The systematic difference between the hand tapper and the impersonal micrometer that has been applied is $0''\cdot45$. Referring back to the impersonal micrometer, which is now generally used, we obtain

$$E = -0''\cdot73 + 0''\cdot28T$$
$$= -0^s\cdot048 + 0^s\cdot019T$$
$$\pm \cdot002 \pm \cdot009.$$

T is measured in centuries from 1900. The correction for 1900 is in close agreement with other determinations. The correction to the motion of the equinox is known to be small; some determinations have given a

* *B.A.N.*, 4, 21, 1927 (see p. 35).

positive correction, others a negative. The correction to the motion here derived is about twice as great as its probable error. The balance of the evidence provided by other determinations is in favour of a small positive correction to Newcomb's motion.

Venus.—We have obtained

$$\Delta l_2 - \Delta L = +0''{\cdot}98 + 2''{\cdot}70T + 0''{\cdot}77T^2 + {\cdot}0471B$$
$$\pm \quad {\cdot}04 \pm \quad {\cdot}16.$$

The discussion has been based on right ascensions of Venus. For ΔL we may therefore use the value of $\Delta\alpha$ obtained above, giving

$$\Delta l_2 = +2''{\cdot}26 + 5''{\cdot}39T + 2''{\cdot}00T^2 + {\cdot}112B.$$

Mercury.—The following representations of V and W have been obtained :—

$$V = +5''{\cdot}81 + 16''{\cdot}01T + 6''{\cdot}35T^2 + {\cdot}385B,$$
$$W = +2''{\cdot}74 + \ 6''{\cdot}82T + 2''{\cdot}42T^2 + {\cdot}147B.$$

We may express V and W in the forms

$$V = 1{\cdot}487\Delta l_1 - 1{\cdot}01\Delta L + \delta V_0 + T\delta V_1,$$
$$W = 0{\cdot}716\Delta l_1 - 0{\cdot}97\Delta L + \Delta W_0 + T\delta W_1,$$

where the last two terms take account of corrections to other elements of the orbits of Mercury and the Earth.

Using the expressions for ΔL, Δl_1, given in (1) and (2) we obtain

$$1{\cdot}487(a' + b'T) + \delta V_0 + T\delta V_1 = +6{\cdot}82 + 19{\cdot}01T. \quad (7)$$

$${\cdot}716(a' + b'T) + \delta W_0 + T\delta W_1 = +3{\cdot}71 + \ 9{\cdot}70T. \quad (8)$$

It is not possible to derive a', b', δV_0, δV_1, δW_0, δW_1 separately from these equations. In terms of the corrections to the elements e, π and e', π' of the orbits of Mercury and the Sun respectively, we have

$$\delta V_0 = -0{\cdot}487\delta\pi - 1{\cdot}137\delta e + 1{\cdot}19e'\delta\pi' + 1{\cdot}58\delta e',$$
$$\delta W_0 = +0{\cdot}284\delta\pi + 0{\cdot}896\delta e - 1{\cdot}11e'\delta\pi' - 1{\cdot}62\delta e'.$$

δV_1, δW_1 have similar expressions with $D_t\delta\pi$ for $\delta\pi$, etc.

The principal portions of the right-hand sides of (7) and (8) arise from the terms involving a' and b', the part contributed by the corrections to the orbital elements being relatively small. It will be seen that the signs of corresponding terms in δV_0 and δW_0 are opposite and that the coefficients are such that $3\delta V_0 + 4\delta W_0$ and $3\delta V_1 + 4\delta W_1$ are small. We therefore assume that we may neglect these quantities, in which the small contributions of the orbital element corrections are much reduced. We then obtain

$$7{\cdot}325(a' + b'T) = +35{\cdot}30 + 95{\cdot}83T,$$

whence

$$a' = +4''{\cdot}82, \qquad b' = +13''{\cdot}08.$$

The correction required by Newcomb's Tables of Mercury is therefore

$$\Delta l_1 = +4''{\cdot}96 + 13''{\cdot}08T + 5''{\cdot}10T^2 + {\cdot}310B.$$

We thus obtain as the corrections required to the mean longitudes of the Sun, Mercury and Venus, respectively, as given by Newcomb's Tables:

$$\Delta L = +1''{\cdot}00 + 2''{\cdot}97T + 1''{\cdot}23T^2 + {\cdot}0747B,$$
$$\Delta l_1 = +4''{\cdot}96 + 13''{\cdot}08T + 5''{\cdot}10T^2 + {\cdot}310B,$$
$$\Delta l_2 = +2''{\cdot}26 + 5''{\cdot}39T + 2''{\cdot}00T^2 + {\cdot}112B,$$

B denoting the fluctuation in the mean longitude of the Moon, as defined in section 2. T is expressed in centuries measured from 1900·0.

Summary.—The fluctuations in the mean longitudes of the Sun, Moon, Mercury and Venus are proportional to the respective mean motions of these bodies and can be attributed to variations in the rotation of the Earth, caused by changes in its moment of inertia. The secular accelerations of the Sun, Mercury and Venus are proportional to their mean motions and can be accounted for by retardation of the Earth's rotation by tidal friction. The secular acceleration of the Moon cannot be predicted theoretically, because tidal friction causes changes in the orbit of the Moon that are too small to be determined by observation.

The observations of the Sun and of Mercury agree in indicating that the average effects of tidal friction during the past two hundred and fifty years are smaller than the average effects during the past two thousand years. There is no indication of any large or sudden change.

Corrections to the mean longitudes of the Sun, Mercury and Venus, involving the fluctuation of the Moon's longitude, are derived. These expressions can be used for the accurate prediction of eclipses and transits of Mercury and Venus, when the tabular error of the Moon's longitude is known.